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Quasideterminant solutions of an integrable chiral model in two dimensions

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Abstract

The Darboux transformation is used to obtain multisoliton solutions of the chiral model in two dimensions. The matrix solutions of the principal chiral model and its Lax pair are expressed in terms of quasideterminants. The iteration of the Darboux transformation gives the quasideterminant multisoliton solutions of the model. It has been shown that the quasideterminant multisoliton solution of the chiral model is the same as obtained by Zakharov and Mikhailov using the dressing method based on the matrix Riemann–Hilbert problem.

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1. Introduction

The principal chiral model (chiral field taking values in a Lie group) is a well-known example of integrable models of relativistically invariant Lagrangian field theories in two dimensions [1–5]. The principal chiral models belong to a more general family of two-dimensional integrable field theories, known as symmetric space sigma models, where the fundamental fields take values in symmetric spaces as their target spaces. The soliton solutions of various sigma models have been obtained using the inverse scattering method, and the multisoliton solutions are obtained by means of Darboux–Backlund transformations [1–11]. In this paper, we study the Darboux transformation of the principal chiral model based on some Lie group and express their soliton solutions in terms of quasideterminants. We show that the matrix solutions of the principal chiral model and those of its associated Lax pair are expressed in terms of quasideterminants. The Darboux transformation also leads to the quasideterminant expressions of the conserved currents of the chiral field. We also obtain the quasideterminant multisoliton solutions of the chiral model from the K times iteration of the Darboux transformations and relate the quasideterminant multisoliton solutions of the chiral field with the well-known solutions of Zakharov and Mikhailov [4] obtained by the matrix Riemann–Hilbert problem. At the end, we discuss the solution of the chiral model based on the Lie group $SU(2)$. We also study the asymptotic behaviour of the solution.

The principal chiral field $g(x)$ with values in some Lie group $\mathcal{G}$ is governed by the Lagrangian[1]

$$L = \frac{1}{2}\mathrm{Tr}(\partial_+ g^{-1}\partial_- g), \tag{1.1}$$

with $g^{-1}g = gg^{-1} = I$. The $\mathcal{G}$-valued field $g(x^+, x^-)$ can be expressed as

$$g(x^+, x^-) \equiv \mathrm{e}^{\mathrm{i}\pi_a T^a} = 1 + \mathrm{i}\pi_a T^a + \frac{1}{2}(\mathrm{i}\pi_a T^a)^2 + \cdots, \tag{1.2}$$

where π_a is in the Lie algebra $\mathbf{g}$ of the Lie group $\mathcal{G}$ and T^a, $a = 1, 2, 3, \ldots, \dim\mathbf{g}$, are anti-Hermitian matrices with normalization $\mathrm{Tr}(T^a T^b) = -\delta^{ab}$ and are the generators of $\mathcal{G}$ in the fundamental representation satisfying

$$[T^a, T^b] = f^{abc}T^c, \tag{1.3}$$

where f^{abc} are the structure constants of the Lie algebra $\mathbf{g}$. For any $X \in \mathbf{g}$, we write $X = X^a T^a$ and $X^a = -\mathrm{Tr}(T^a X)$. The action (1.1) is invariant under a global continuous symmetry

$$\mathcal{G}_L \times \mathcal{G}_R : g(x^+, x^-) \longmapsto UgV^{-1}, \tag{1.4}$$

where $U \in \mathcal{G}_L$ and $V \in \mathcal{G}_R$. The Noether conserved current associated with the $\mathcal{G}_R$ transformation is $j_\pm = -g^{-1}\partial_\pm g$, which takes values in the Lie algebra $\mathbf{g}$, so that one can decompose the current into components $j_\pm(x^+, x^-) = j^a_\pm(x^+, x^-)T^a$. The conserved current corresponding to the $\mathcal{G}_L$ transformation is $-gj_\pm g^{-1}$. The equation of motion of the principal chiral model is the conservation equation and the zero curvature condition

$$\partial_+ j_- + \partial_- j_+ = 0, \tag{1.5}$$

$$\partial_- j_+ - \partial_+ j_- + [j_+, j_-] = 0. \tag{1.6}$$

The equations of motion (1.5) and (1.6) can be written as the compatibility condition of the following Lax pair,

$$\partial_+ V(\lambda) = \frac{1}{1-\lambda} j_+ V(\lambda), \tag{1.7}$$

$$\partial_- V(\lambda) = \frac{1}{1+\lambda} j_- V(\lambda), \tag{1.8}$$

where λ is a real (or complex) parameter and V is an invertible $N \times N$ matrix, in general. We solve the Lax pair to find the matrix solution $V(\lambda)$ such that $V(0) = g$. If we have any collection $(V(\lambda), j_\pm)$ which solves the Lax pair (1.7) and (1.8), then $V(0) = g$ solves the chiral field equation (1.5). In the following section, we define the Darboux transformation via a Darboux matrix on matrix solutions V of the Lax pair (1.7) and (1.8). To write down the explicit expressions for matrix solutions of the chiral model, we will use the notion of quasideterminant introduced by Gelfand and Retakh [21–25].

Let X be an $N \times N$ matrix over a ring R (noncommutative, in general). For any $1 \leqslant i, j \leqslant N$, let r_i be the ith row and c_j be the jth column of X. There exist N^2 quasideterminants denoted by $|X|_{ij}$ for $i, j = 1, \ldots, N$ and are defined by

$$|X|_{ij} = \begin{vmatrix} X^{ij} & c^i_j \\ r^j_i & \boxed{x_{ij}} \end{vmatrix} = x_{ij} - r^j_i (X^{ij})^{-1} c^i_j, \tag{1.9}$$

where x_{ij} is the ijth entry of X, r^j_i represents the ith row of X without the jth entry, c^i_j represents the jth column of X without the ith entry and X^{ij} is the submatrix of X obtained

[1] The spacetime conventions are such that the light-cone coordinates $x^\pm$ are related to the orthonormal coordinates by $x^\pm = \frac{1}{2}(t \pm x)$ with derivatives $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$.

by removing from X the ith row and the jth column. The quasideterminants are also denoted by the following notation. If the ring R is commutative, i.e., the entries of the matrix X all commute, then

$$|X|_{ij} = (-1)^{i+j}\frac{\det X}{\det X^{ij}}. \tag{1.10}$$

For a detailed account of quasideterminants and their properties see, e.g., [21–25]. In this paper, we will consider only quasideterminants that are expanded about an $N \times N$ matrix over a commutative ring. Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a block decomposition of any $K \times K$ matrix where the matrix D is $N \times N$ and A is invertible. The ring R in this case is the (noncommutative) ring of $N \times N$ matrices over another commutative ring. The quasideterminant of the $K \times K$ matrix expanded about the $N \times N$ matrix D is defined by

$$\begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} = D - CA^{-1}B. \tag{1.11}$$

An important property of quasideterminants is the noncommutative Jacobi identity. For a general quasideterminant expanded about an $N \times N$ matrix D, we have

$$\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} = \begin{vmatrix} E & G \\ J & \boxed{D} \end{vmatrix} - \begin{vmatrix} E & F \\ J & \boxed{C} \end{vmatrix}\begin{vmatrix} E & F \\ H & \boxed{A} \end{vmatrix}^{-1}\begin{vmatrix} E & G \\ H & \boxed{B} \end{vmatrix}. \tag{1.12}$$

From the noncommutative Jacobi identity, we get the homological relation

$$\begin{vmatrix} E & F & G \\ H & A & \boxed{B} \\ J & C & D \end{vmatrix} = \begin{vmatrix} E & F & O \\ H & A & \boxed{O} \\ J & C & I \end{vmatrix}\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix}, \tag{1.13}$$

where O and I denote the null and identity matrices, respectively. The quasideterminants have found various applications in the theory of integrable systems, where the multisoliton solutions of various noncommutative integrable systems are expressed in terms of quasideterminants (see, e.g., [26–33]).

2. Darboux transformation

The Darboux transformation is one of the well-known methods of obtaining multisoliton solutions of integrable systems [18–20]. We define the Darboux transformation on the matrix solutions of the Lax pair (1.7) and (1.8), in terms of an $N \times N$ matrix $D(x^+, x^-, \lambda)$, called the Darboux matrix. For a general discussion on the Darboux matrix approach see, e.g., [12–17]. The Darboux matrix relates the two matrix solutions of the Lax pair (1.7) and (1.8) in such a way that the Lax pair is covariant under the Darboux transformation. The Darboux transformation on the matrix solution of the Lax pair (1.7) and (1.8) is defined by

$$\widetilde{V}(\lambda) = D(x^+, x^-, \lambda)V(\lambda). \tag{2.1}$$

For the Lax pair (1.7) and (1.8) to be covariant under the Darboux transformation (2.1), we require

$$\partial_+ \widetilde{V}(\lambda) = \frac{1}{1-\lambda} \widetilde{j}_+ \widetilde{V}(\lambda), \tag{2.2}$$

$$\partial_- \widetilde{V}(\lambda) = \frac{1}{1+\lambda} \widetilde{j}_- \widetilde{V}(\lambda). \tag{2.3}$$

By substituting equation (2.1) into equations (2.2) and (2.3), we get the following condition on the Darboux matrix $D(\lambda)$:

$$\partial_\pm D(\lambda) V(\lambda) + D(\lambda) \frac{1}{1\mp\lambda} j_\pm V(\lambda) = \frac{1}{1\mp\lambda} \widetilde{j}_\pm D(\lambda) V(\lambda). \tag{2.4}$$

For our system, we make the following ansatz for the Darboux matrix,

$$D(x^+, x^-, \lambda) = \lambda I - S(x^+, x^-), \tag{2.5}$$

where $S(x^+, x^-)$ is some $N \times N$ matrix to be determined and I is an $N \times N$ identity matrix. Note that we consider here the Darboux matrix of degree 1 which is linear in λ. Therefore, to construct the Darboux matrix $D(x^+, x^-, \lambda)$, it is only necessary to determine the matrix $S(x^+, x^-)$. Now substituting (2.1) into equation (2.4) and using (1.7) and (1.8), we get the following Darboux transformation for the Lie algebra valued conserved currents $\widetilde{j}_\pm$:

$$\begin{aligned} \widetilde{j}_+ &= j_+ + \partial_+ S, \\ \widetilde{j}_- &= j_- - \partial_- S, \end{aligned} \tag{2.6}$$

and the matrix S is subjected to satisfy the following conditions:

$$\partial_+ S(I - S) = [j_+, S], \tag{2.7}$$

$$\partial_- S(I + S) = [j_-, S]. \tag{2.8}$$

These new transformed currents are also conserved and curvature free, i.e.,

$$\partial_+ \widetilde{j}_- + \partial_- \widetilde{j}_+ = 0, \tag{2.9}$$

$$\partial_- \widetilde{j}_+ - \partial_+ \widetilde{j}_- + [\widetilde{j}_+, \widetilde{j}_-] = 0. \tag{2.10}$$

Now we proceed to determine the matrix S so that the explicit Darboux transformation in terms of particular solutions of the Lax pair can be constructed.

Let $\lambda_1, \ldots, \lambda_N$, be N distinct real (or complex) constant parameters and $\lambda_i \neq \pm 1$; $i = 1, 2, \ldots, N$. Let us also define N constant column vectors $|1\rangle, |2\rangle, \ldots, |N\rangle$, such that

$$M = (V(\lambda_1)|1\rangle, \ldots, V(\lambda_N)|N\rangle) = (|m_1\rangle, \ldots, |m_N\rangle) \tag{2.11}$$

is an invertible $N \times N$ matrix. Each column $|m_i\rangle = V(\lambda_i)|i\rangle$ in M is a column solution of the Lax pair (1.7) and (1.8) when $\lambda = \lambda_i$. That is, it satisfies

$$\partial_+ |m_i\rangle = \frac{1}{1-\lambda_i} j_+ |m_i\rangle, \tag{2.12}$$

$$\partial_- |m_i\rangle = \frac{1}{1+\lambda_i} j_- |m_i\rangle, \tag{2.13}$$

and $i = 1, 2, \ldots, N$. If we define an $N \times N$ matrix of particular eigenvalues as

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N), \tag{2.14}$$

then the Lax pair (2.12) and (2.13) can be written in the $N \times N$ matrix form as

$$\partial_+ M = j_+ M(I - \Lambda)^{-1}, \tag{2.15}$$

$$\partial_- M = j_- M(I + \Lambda)^{-1}, \tag{2.16}$$

where the $N \times N$ matrix M is a particular matrix solution of the Lax pair (1.7) and (1.8) with Λ being a matrix of particular eigenvalues. In terms of particular matrix solution M of the Lax pair (1.7) and (1.8), we define the matrix S as

$$S = M\Lambda M^{-1}. \tag{2.17}$$

Now we show that the matrix S defined in (2.17), satisfies equations (2.7) and (2.8). First, we take the x^+ derivative of the matrix (2.17) so that we have

$$\begin{aligned}\partial_+ S &= \partial_+(M\Lambda M^{-1}),\\ &= \partial_+ M\Lambda M^{-1} + M\Lambda\partial_+(M^{-1}),\\ &= j_+ M(I-\Lambda)^{-1}\Lambda M^{-1} - M\Lambda M^{-1} j_+ M(I-\Lambda)^{-1}M^{-1},\\ &= -j_+ + M(I-\Lambda)M^{-1} j_+ M(I-\Lambda)^{-1}M^{-1},\\ &= -j_+ + (I-S)j_+(I-S)^{-1},\end{aligned} \tag{2.18}$$

which is equation (2.7). Similarly, operating ∂_- on (2.17), we get

$$\partial_- S = j_- - (I+S)j_-(I+S)^{-1}, \tag{2.19}$$

which is nothing but equation (2.8). This shows that the choice (2.17) of the matrix S satisfies all the conditions imposed by the covariance of the Lax pair under the Darboux transformation. Therefore, we say that the transformation,

$$\begin{aligned}\widetilde{V} &= (\lambda I - M\Lambda M^{-1})V,\\ \widetilde{j}_\pm &= M(I \mp \Lambda)M^{-1} j_\pm M(I \mp \Lambda)^{-1}M^{-1},\end{aligned} \tag{2.20}$$

is the required Darboux transformation of the chiral model in terms of the particular matrix solution M with the particular eigenvalue matrix Λ. Let us now introduce a primitive field $F_\pm$ such that $j_\pm = F_\pm F_\pm^{-1}$, which transforms in a simple way under the Darboux transformation, i.e.,

$$\tilde{F}_\pm = M(I \mp \Lambda)M^{-1}F_\pm. \tag{2.21}$$

The Darboux transformation on the chiral field $g(x)$ is now defined by

$$\widetilde{g} = \widetilde{V}(0) = -(M\Lambda M^{-1})g. \tag{2.22}$$

Since we have assumed M to be invertible, therefore, we require that $\det M \neq 0$. At this stage, we conclude that if the collection $(V, j_\pm)$ is a solution of the Lax pair (1.7) and (1.8) and the matrix S is defined by (2.17), then $(\widetilde{V}, \widetilde{j}_\pm)$ defined by (2.20) by means of Darboux transformation (2.5) is also a solution of the same Lax pair. This establishes the covariance of the Lax pair (1.7) and (1.8) under the Darboux transformation (2.5).

If the chiral fields take values in the Lie group $U(N)$, then we also require for the new solutions to take values in $U(N)$. We know that the Lie group $U(N)$ consists of all $N \times N$ matrices g such that $g^\dagger = g^{-1}$. An arbitrary matrix X belongs to the Lie algebra $\mathbf{u}(N)$ of the Lie group $U(N)$ if and only if $X^\dagger = -X$. Since the currents $j_\pm$ are $\mathbf{u}(N)$ valued, therefore, we require that the new currents $\widetilde{j}_\pm$ obtained by the Darboux transformation must be $\mathbf{u}(N)$ valued, i.e., they must be anti-Hermitian. This leads to the following condition on the matrix S:

$$\partial_\pm(S + S^\dagger) = 0. \tag{2.23}$$

For the matrix S to satisfy (2.23), we proceed by taking specific values of parameters λ_i. Let μ be a non-zero complex number and $\lambda_i = \mu (i = 1, 2, \ldots, N)$. Now choose $|i\rangle$ such that

$$\langle m_i | m_j \rangle = 0 \qquad \text{for} \qquad \lambda_i \neq \lambda_j \tag{2.24}$$

holds everywhere, and $|m_i\rangle$ are all linearly independent. From the definition of the matrix S, it can be observed that

$$\langle m_i | (S^\dagger + S) | m_j \rangle = (\bar{\lambda}_i + \lambda_j) \langle m_i | m_j \rangle, \tag{2.25}$$

implying that $\langle m_i | m_j \rangle = 0$, when $\lambda_i \neq \lambda_j$. If $\lambda_i = \lambda_j = \mu$, we have

$$\langle m_i | (S^\dagger + S) | m_j \rangle = \langle m_i | (\mu + \bar{\mu}) | m_j \rangle. \tag{2.26}$$

Since $|m_i\rangle$'s are all linearly independent, therefore, equation (2.26) implies

$$(S^\dagger + S) = (\mu + \bar{\mu}) I, \tag{2.27}$$

which further implies (2.23). Again from the Lax pair (1.7) and (1.8), we have

$$\langle m_i | S^\dagger S | m_j \rangle = \langle m_i | \bar{\lambda}_i \lambda_j | m_j \rangle,$$

thus, if $\lambda_i = \mu$,

$$S^\dagger S = \bar{\mu} \mu. \tag{2.28}$$

For the Lie group $SU(N)$, we have to impose further condition on the new conserved currents $\widetilde{j}_\pm$. We know that the Lie group $SU(N)$ consists of all $N \times N$ matrices g such that $g \in U(N)$ and $\det g = 1$. An arbitrary matrix X belongs to the Lie algebra $\mathbf{su}(N)$ of the group $SU(N)$, if and only if $\operatorname{Tr} X = 0$. So if the chiral field $g(x)$ takes values in $SU(N)$, then we also require that $\operatorname{Tr} j_\pm = 0$, $\operatorname{Tr} \widetilde{j}_\pm = 0$; and for this to be the case, the matrix S is required to satisfy

$$\operatorname{Tr} \partial_\pm S = 0. \tag{2.29}$$

The condition $\operatorname{Tr} \partial_\pm S = 0$ is satisfied by equations (2.18) and (2.19), using the cyclicity of trace.

We impose the reality condition on solutions $V(\lambda)$ of the Lax pair (1.7) and (1.8)

$$V^\dagger(\bar{\lambda}) V(\lambda) = V^\dagger(\bar{\lambda}) V(\lambda) \in \operatorname{Span}\{I\}, \tag{2.30}$$

where I is an $N \times N$ unit matrix and $\operatorname{Span}\{I\}$ is the subspace of the underlying Lie group spanned by I. To obtain well-defined transformed solutions, the Darboux transformation must preserve this reality condition, i.e.,

$$\widetilde{V}^\dagger(\bar{\lambda}) \widetilde{V}(\lambda) \in \operatorname{Span}\{I\}. \tag{2.31}$$

Using (2.1) and (2.5), also making use of (2.28) and (2.27), we see that

$$\widetilde{V}^\dagger(\bar{\lambda}) \widetilde{V}(\lambda) = (\lambda^2 - \lambda(\mu + \bar{\mu}) + \mu\bar{\mu}) V^\dagger(\bar{\lambda}) V(\lambda) \in \operatorname{Span}\{I\},$$

i.e., the transformed solution satisfies the reality condition, or in other words, the Darboux transformation preserves the reality condition (2.31). In the following section, we will express the solutions of the chiral model obtained by the Darboux transformation in terms of quasideterminants that are expanded about an $N \times N$ matrix over a commutative ring.

3. Quasideterminant solutions

Since the particular solution M of the Lax pair (1.7) and (1.8) is an invertible $N \times N$ matrix, therefore we can express the Darboux transformations (2.20) and (2.22) in terms of quasideterminants. The Darboux transformed solution $\widetilde{V}$ of the Lax pair (1.7) and (1.8) is expressed as

$$\begin{aligned}\widetilde{V} &= (\lambda I - S)V = (\lambda I - M\Lambda M^{-1})V,\\ &= \begin{vmatrix} M & I \\ M\Lambda & \boxed{\lambda I} \end{vmatrix} V,\end{aligned} \tag{3.1}$$

and the chiral field $\widetilde{g}$ is expressed as

$$\widetilde{g} = \widetilde{V}(0) = -Sg = -(M\Lambda M^{-1})g = \begin{vmatrix} M & I \\ M\Lambda & \boxed{O} \end{vmatrix} g. \tag{3.2}$$

Similarly from (2.21) the conserved currents $\tilde{j}_\pm$ are expressed as

$$\tilde{j}_\pm = \tilde{F}_\pm \tilde{F}_\pm^{-1} = \begin{vmatrix} M & I \\ M(I \mp \Lambda) & \boxed{O} \end{vmatrix} j_\pm \begin{vmatrix} M & I \\ M(I \mp \Lambda) & \boxed{O} \end{vmatrix}^{-1}. \tag{3.3}$$

For the next iteration of the Darboux transformation, we take M_1, M_2 to be two particular solutions of the Lax pair (2.15) and (2.16) at $\Lambda = \Lambda_1$ and $\Lambda = \Lambda_2$, respectively. Using the notation $V[1] = V$, $g[1] = g$, $j_\pm[1] = j_\pm$, $F_\pm[1] = F_\pm$ and $V[2] = \widetilde{V}$, $g[2] = \widetilde{g}$, $j_\pm[2] = \tilde{j}_\pm$, $F_\pm[2] = \tilde{F}_\pm$, we write the two-fold Darboux transformation on V as

$$V[3] = (\lambda I - S[2])\,(\lambda I - S[1])\,V = (\lambda I - S[2])V[2],$$

where $S[1] = M_1\Lambda_1 M_1^{-1}$. By writing $S[2] = M[2]\Lambda_2 M[2]^{-1}$, we get

$$V[3] = (\lambda I - M[2]\Lambda_2 M[2]^{-1})V[2], \tag{3.4}$$

where $M[2] = V[2]|_{V\to M_2}$, so that after the action of $\lambda I - S[1]$, the vector $\left|m_j^{(2)}\right\rangle$ transforms as $\left(\lambda_j^{(2)} I - S[1]\right)\left|m_j^{(2)}\right\rangle$. Therefore, we have

$$M[2] = (M_2\Lambda_2 - S[1]M_2) = \begin{vmatrix} M_1 & M_2 \\ M_1\Lambda_1 & \boxed{M_2\Lambda_2} \end{vmatrix}. \tag{3.5}$$

By using equations (3.1) and (3.5) in (3.4), we get

$$\begin{aligned} V[3] &= \lambda \begin{vmatrix} M_1 & I \\ M_1\Lambda_1 & \boxed{\lambda I} \end{vmatrix} V - \begin{vmatrix} M_1 & M_2 \\ M_1\Lambda_1 & \boxed{M_2\Lambda_2} \end{vmatrix} \Lambda_2 \begin{vmatrix} M_1 & M_2 \\ M_1\Lambda_1 & \boxed{M_2\Lambda_2} \end{vmatrix}^{-1} \begin{vmatrix} M_1 & I \\ M_1\Lambda_1 & \boxed{\lambda I} \end{vmatrix} V,\\ &= \begin{vmatrix} M_1\Lambda_1 & \lambda V \\ M_1\Lambda_1^2 & \boxed{\lambda^2 V} \end{vmatrix} - \begin{vmatrix} M_1\Lambda_1 & M_2\Lambda_2 \\ M_1\Lambda_1^2 & \boxed{M_2\Lambda_2^2} \end{vmatrix} \begin{vmatrix} M_1\Lambda_1 & M_2\Lambda_2 \\ M_1 & \boxed{M_2} \end{vmatrix}^{-1} \begin{vmatrix} M_1\Lambda_1 & \lambda V \\ M_1 & \boxed{V} \end{vmatrix},\\ &= \begin{vmatrix} M_1 & M_2 & I \\ M_1\Lambda_1 & M_2\Lambda_2 & \lambda I \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \boxed{\lambda^2 I} \end{vmatrix} V, \end{aligned} \tag{3.6}$$

where we have used homological relation (1.13) in the second step and the noncommutative Jacobi identity (1.12) in the last step.

Similarly, the two-fold Darboux transformation on conserved currents $j_\pm$ gives

$$j_\pm[3] = F_\pm[3]F_\pm[3]^{-1}, \tag{3.7}$$

where the factor $F_\pm[3]$ is expressed in terms of quasideterminants as

$$\begin{aligned}
F_\pm[3] &= (M[2](I\mp\Lambda_2)M[2]^{-1})(M[1](I\mp\Lambda_1)M[1]^{-1})F_\pm \\
&= -\begin{vmatrix} M_1 & M_2 \\ M_1\Lambda_1 & \boxed{M_2\Lambda_2}\end{vmatrix}(I\mp\Lambda_2)\begin{vmatrix} M_1 & M_2 \\ M_1\Lambda_1 & \boxed{M_2\Lambda_2}\end{vmatrix}^{-1}\begin{vmatrix} M_1 & I \\ M_1(I\mp\Lambda_1) & \boxed{O}\end{vmatrix}F_\pm \\
&= -\begin{vmatrix} M_1 & M_2 \\ M_1(I\mp\Lambda_1) & \boxed{M_2(I\mp\Lambda_2)}\end{vmatrix}(I\mp\Lambda_2) \\
&\quad\times\begin{vmatrix} M_1 & M_2 \\ M_1(I\mp\Lambda_1) & \boxed{M_2(I\mp\Lambda_2)}\end{vmatrix}^{-1}\begin{vmatrix} M_1 & I \\ M_1(I\mp\Lambda_1) & \boxed{O}\end{vmatrix}F_\pm \\
&= -\begin{vmatrix} M_1 & M_2(I\mp\Lambda_2) \\ M_1(I\mp\Lambda_1) & \boxed{M_2(I\mp\Lambda_2)^2}\end{vmatrix}\begin{vmatrix} M_1 & M_2 \\ M_1(I\mp\Lambda_1) & \boxed{M_2(I\mp\Lambda_2)}\end{vmatrix}^{-1} \\
&\quad\times\begin{vmatrix} M_1 & I \\ M_1(I\mp\Lambda_1) & \boxed{O}\end{vmatrix}F_\pm \\
&= \begin{vmatrix} M_1 & M_2 & I \\ M_1(I\mp\Lambda_1) & M_2(I\mp\Lambda_2) & O \\ M_1(I\mp\Lambda_1)^2 & M_2(I\mp\Lambda_2)^2 & \boxed{O}\end{vmatrix}F_\pm,
\end{aligned}$$

where we have used the homological relation (1.13) and noncommutative Jacobi identity (1.12) for obtaining the last step.

We can iterate the Darboux transformation K times and obtain the quasideterminant multisoliton solution of the chiral model. For each $k = 1, 2, \ldots, K$, let M_k be an invertible $N \times N$ matrix solution of the Lax pair (1.7) and (1.8) at $\Lambda = \Lambda_k$, then the Kth solution $V[K+1]$ is expressed as

$$\begin{aligned}
V[K+1] &= \prod_{k=1}^{K}(\lambda I - S[K-k+1])V = \prod_{k=1}^{K}\begin{vmatrix} M[K-k+1] & I \\ M[K-k+1]\Lambda_{K-k+1} & \boxed{\lambda I}\end{vmatrix}V, \\
&= \lambda V[K] - M[K]\Lambda_K M[K]^{-1}V[K], \\
&= \begin{vmatrix} M_1 & M_2 & \cdots & M_K & I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & \lambda I \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & \lambda^2 I \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{\lambda^K I}\end{vmatrix}V.
\end{aligned}\tag{3.8}$$

The above results can be proved by induction using the properties of quasideterminants. First, we see that the result (3.8) is true for $K = 1$ and gives equation (3.1) directly. Next, we consider

$$\begin{aligned}
V[K+2] &= (\lambda I - S[K+1])V[K+1], \\
&= \lambda V[K+1] - S[K+1]V[K+1], \\
&= \lambda V[K+1] - M[K+1]\Lambda_{K+1}M[K+1]^{-1}V[K+1].
\end{aligned}\tag{3.9}$$

By using equation (3.8) in the expression (3.9) and using the fact that $M[i] = V[i]|_{V\to M_i}$, we get

$$V[K+2] = \begin{vmatrix} M_1 & M_2 & \cdots & M_K & \lambda I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & \lambda^2 I \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{\lambda^K I} \end{vmatrix} V$$

$$- \begin{vmatrix} M_1 & M_2 & \cdots & M_K & M_{K+1} \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & M_{K+1}\Lambda_{K+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{M_{K+1}\Lambda_{K+1}^K} \end{vmatrix} \Lambda_{K+1}$$

$$\times \begin{vmatrix} M_1 & M_2 & \cdots & M_K & M_{K+1} \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & M_{K+1}\Lambda_{K+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{M_{K+1}\Lambda_{K+1}^K} \end{vmatrix}^{-1}$$

$$\times \begin{vmatrix} M_1 & M_2 & \cdots & M_K & \lambda I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & \lambda^2 I \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{\lambda^K I} \end{vmatrix} V.$$

Now rearranging the above expression and using the noncommutative Jacobi identity (1.12) and homological relations (1.13), we get

$$V[K+2] = \begin{vmatrix} M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & \lambda V \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & \lambda^2 V \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^{K+1} & M_2\Lambda_2^{K+1} & \cdots & M_K\Lambda_K^{K+1} & \boxed{\lambda^K V} \end{vmatrix}$$

$$- \begin{vmatrix} M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & M_{K+1}\Lambda_{K+1} \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & M_{K+1}\Lambda_{K+1}^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^{K+1} & M_2\Lambda_2^{K+1} & \cdots & M_K\Lambda_K^{K+1} & \boxed{M_{K+1}\Lambda_{K+1}^{K+1}} \end{vmatrix}$$

$$\times \begin{vmatrix} M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & M_{K+1}\Lambda_{K+1} \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & M_{K+1}\Lambda_{K+1}^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & M_{K+1}\Lambda_{K+1}^K \\ M_1 & M_2 & \cdots & M_K & \boxed{M_{K+1}} \end{vmatrix}^{-1}$$

$$\times \begin{vmatrix} M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & \lambda V \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & \lambda^2 V \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \lambda^K V \\ M_1 & M_2 & \cdots & M_K & \boxed{V} \end{vmatrix},$$

$$= \begin{vmatrix} M_1 & M_2 & \cdots & M_{K+1} & I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_{K+1}\Lambda_{K+1} & \lambda I \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_{K+1}\Lambda_{K+1}^2 & \lambda^2 I \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^{K+1} & M_2\Lambda_2^{K+1} & \cdots & M_{K+1}\Lambda_{K+1}^{K+1} & \boxed{\lambda^{K+1} I} \end{vmatrix} V. \tag{3.10}$$

Therefore (3.8) is verified. The multisoliton solution $g[K+1]$ of the chiral model can be readily obtained by taking $\lambda = 0$ in the expression of $V[K+1]$, i.e.,

$$g[K+1] = \prod_{k=1}^{K}(-1)^k S[K-k+1]g = \prod_{k=1}^{K} \begin{vmatrix} M[K-k+1] & I \\ M[K-k+1]\Lambda_{K-k+1} & \boxed{O} \end{vmatrix} g,$$

$$= \begin{vmatrix} M_1 & M_2 & \cdots & M_K & I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & O \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & O \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{O} \end{vmatrix} g. \tag{3.11}$$

Similarly, the K times iteration of the Darboux transformation gives the following expression of the conserved currents,

$$j_\pm[K+1] = F_\pm[K+1]F_\pm[K+1]^{-1}, \tag{3.12}$$

where

$$F_\pm[K+1] = (I \mp S[K])\cdots(I \mp S[2])(I \mp S[1])F_\pm,$$

$$\times \prod_{k=1}^{K} \begin{vmatrix} M[K-k+1] & I \\ M[K-k+1](I \mp \Lambda_{K-k+1}) & \boxed{O} \end{vmatrix} F_\pm, \tag{3.13}$$

$$= \begin{vmatrix} M_1 & M_2 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & M_2(I \mp \Lambda_2) & \cdots & M_K(I \mp \Lambda_K) & O \\ M_1(I \mp \Lambda_1)^2 & M_2(I \mp \Lambda_2)^2 & \cdots & M_K(I \mp \Lambda_K)^2 & O \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & M_2(I \mp \Lambda_2)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix} F_\pm. \tag{3.14}$$

The expression (3.14) can also be proved by induction in the same way as we did for (3.8). Hence, we see that equations (3.13) and (3.14) together with (3.12) are the required expressions of Kth conserved currents of the chiral model expressed in terms of quasideterminants involving particular solutions of the linear problem associated with the chiral model. Note that equations (3.8) and (3.11) are the required quasideterminant expressions for the Kth iteration of V and g.

4. Relation with Zakharov–Mikhailov's dressing method

In this section, we relate the quasideterminant multisoliton solutions of the previous section with the solutions obtained by Zakharov and Mikhailov using an equivalent method known as the dressing method. In the original approach of Zakharov and Mikhailov, the analytical properties of the matrix function $\mathcal{D}(\lambda)$ (now referred to as dressing function) are studied in the complex λ-plane. In fact, the dressing function $\mathcal{D}(\lambda)$ in the Zakharov–Mikhailov approach is equivalent to $(\lambda-\mu)^{-1}D(\lambda)$, where $D(\lambda)$ is the Darboux matrix (2.5). Note that equation (2.4) remains invariant, if the Darboux-dressing matrix is multiplied by a scalar factor. In particular, it is required that $\mathcal{D}(\lambda)$ should be meromorphic and $\mathcal{D}(\lambda)\to I$ as $\lambda\to\pm\infty$. In other words, we say that the matrix function $\mathcal{D}(\lambda)$ has a pole at some λ or any entry of $\mathcal{D}(\lambda)$ has a pole at that particular value of λ. If we take the simple case, in which $\mathcal{D}(\lambda)$ has a single pole at $\lambda=\mu$, the dressing function $\mathcal{D}(\lambda)$ is expressed in terms of a Hermitian projector P. In what follows, we show that our Darboux matrix expressed as a quasideterminant can be written in terms of the Hermitian projection operator, resulting in a solution of the chiral model without much use of analytical properties of matrix functions involved.

The Darboux matrix (2.5) can also be written in terms of the projector. For this purpose, we make use of equation (2.25), i.e., we write

$$\begin{aligned} S|m_i\rangle &= \lambda_i|m_i\rangle, && i=1,2,\ldots,n \\ S|m_j\rangle &= \bar{\lambda}_j|m_j\rangle, && j=n+1,n+2,\ldots,N. \end{aligned}$$

Now, we set $\lambda_i=\mu$ and $\lambda_j=\bar{\mu}$ so that the matrix S may be written as

$$S=\mu P+\bar{\mu}P^{\perp}, \tag{4.1}$$

where P is the Hermitian projector, i.e., $P^{\dagger}=P$. Also we have $P^2=P$ and $P^{\perp}=I-P$. The projector P is completely characterized by two subspaces $U=\operatorname{Im}P$ and $W=\operatorname{Ker}P$ given by the condition $P^{\perp}U=0$ and $PW=0$, so that P is defined as a Hermitian projection on a complex subspace and $P^{\perp}=I-P$ as projection on the orthogonal space. Now the matrix S is expressed as

$$\begin{aligned} S &= \mu P+\bar{\mu}(I-P), \\ &= (\mu-\bar{\mu})P+\bar{\mu}I. \end{aligned} \tag{4.2}$$

The Darboux matrix which is expressed as a quasideterminant in the previous section can now be written as

$$\begin{aligned} D(\lambda) &= \begin{vmatrix} M & I \\ M\Lambda & \boxed{\lambda I} \end{vmatrix} \\ &= \lambda I-(\mu-\bar{\mu})P-\bar{\mu}I, \\ &= (\lambda-\bar{\mu})I-(\mu-\bar{\mu})P, \\ &= (\lambda-\bar{\mu})\left(I-\frac{\mu-\bar{\mu}}{\lambda-\bar{\mu}}P\right). \end{aligned} \tag{4.3}$$

In the expression (4.3), the Darboux-dressing function, expressed as quasideterminant containing the particular matrix solution M of the Lax pair (1.7) and (1.8), is shown to be expressed in terms of a Hermitian projector P defined in terms of the particular column solutions $|m_i\rangle$ of the Lax pair. The Darboux-dressing function (4.3) can also be used to obtain the multisoliton solution of the system. For the case of models based on Lie groups of $N\times N$ matrices, we set $\lambda_i=\mu,(i=1,2,\ldots,n)$ and $\lambda_j=\bar{\mu},(j=n+1,\ldots,N)$, so that the solution $V[2]$ is given by

$$V[2] = \left(\lambda I - \mu \sum_{i=1}^{n} \frac{|m_i\rangle\langle m_i|}{\langle m_i|m_i\rangle} - \bar{\mu} \sum_{j=n+1}^{N} \frac{|m_j\rangle\langle m_j|}{\langle m_j|m_j\rangle}\right) V,$$

$$= (\lambda - \bar{\mu}) \left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} \sum_{i=1}^{n} \frac{|m_i\rangle\langle m_i|}{\langle m_i|m_i\rangle}\right) V = (\lambda - \bar{\mu}) \left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} P\right) V.$$

The Kth time iteration then gives the Kth solution $V[K+1]$ of the Lax pair

$$V[K+1] = \prod_{k=1}^{K} \begin{vmatrix} M[K-k+1] & I \\ M[K-k+1]\Lambda_{K-k+1} & \boxed{\lambda I} \end{vmatrix} V,$$

$$= \prod_{k=1}^{K} (\lambda - \bar{\mu}_{K-k+1}) \left(I - \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\lambda - \bar{\mu}_{K-k+1}} P[K-k+1]\right) V, \tag{4.4}$$

with the $(K+1)$th soliton solution $g[K+1]$ of the chiral model given by

$$g[K+1] = \prod_{k=1}^{K} (-\bar{\mu}_{K-k+1}) \left(I + \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\bar{\mu}_{K-k+1}} P[K-k+1]\right) V, \tag{4.5}$$

where the Hermitian projection in this case is

$$P[k] = \sum_{i=1}^{n} \frac{|m_i[k]\rangle\, \langle m_i[k]|}{\langle m_i[k]|\, m_i[k]\rangle}, \qquad k = 1, 2, \ldots, K,$$

with

$$|m_i[k]\rangle = \left(\lambda_i^{(k)} I - S[k-1]\right) \left|m_i^{(k)}\right\rangle,$$

and the kth particular matrix solution M_k of the Lax pair is written in terms of kth particular column solutions as

$$M_k = \left(\left|m_1^{(k)}\right\rangle, \left|m_2^{(k)}\right\rangle, \ldots \left|m_N^{(k)}\right\rangle\right). \tag{4.6}$$

Now the expressions for the transformed currents $j_\pm[K+1]$ are given by

$$j_\pm[K+1] = \prod_{k=1}^{K} \left(I \mp \frac{(\mu_{K-k+1} - \bar{\mu}_{K-k+1})}{(1 \mp \bar{\mu}_{K-k+1})} P[K-k+1]\right) j_\pm \prod_{l=1}^{K} \left(I \mp \frac{(\bar{\mu}_l - \mu_l)}{(1 \mp \mu_l)} P[l]\right). \tag{4.7}$$

The expressions (4.4), (4.5) and (4.7) can also be written as sum of K terms by using the condition that $V[K] = 0$ if $\lambda = \mu_i$, $V = |m_i\rangle$. The method is illustrated in [1] (also see [34] for reference). The final expressions for $V[K+1]$, $g[K+1]$ and $j_\pm[K+1]$ are then given as

$$V[K+1] = \sum_{k,l=1}^{K} (\lambda - \bar{\mu}_k) \left(I - \frac{R_k}{\lambda - \bar{\mu}_k}\right) V, \tag{4.8}$$

$$g[K] = \sum_{k,l=1}^{K} (-\bar{\mu}_k) \left(I + \frac{R_k}{\bar{\mu}_k}\right) g, \tag{4.9}$$

$$j_\pm[K] = \sum_{k=1}^{K} \left(I \mp \frac{R_k}{1 \mp \bar{\mu}_k}\right) j_\pm \sum_{l=1}^{K} \left(I \mp \frac{R_l}{1 \mp \mu_l}\right), \tag{4.10}$$

where the function R_k is defined by

$$R_k = \sum_{l=1}^{K} (\mu_l - \bar{\mu}_k) \sum_{i=1}^{n} \frac{\left|m_i^{(k)}\right\rangle\left\langle m_i^{(l)}\right|}{\left\langle m_i^{(k)}\middle|m_i^{(l)}\right\rangle}. \tag{4.11}$$

By expanding the right-hand side in equation (4.8) and using (2.24), we see that the two expressions for the Kth iteration of V, i.e., equations (3.8) and (4.8) are equivalent.

5. The $SU(2)$ model

In this section, we briefly discuss how to calculate the soliton solution of the principal chiral model based on the Lie group $SU(2)$ using the method outlined in the previous section. For the $SU(2)$ case, the solution has been obtained in [4]. Let us first calculate the one-soliton solution of the chiral model using the dressing (Darboux) matrix (4.3). The matrix solution $V[1]$ of the Lax pair (1.7) and (1.8) is given by

$$\widetilde{V} = (\lambda I - M\Lambda M^{-1})V. \tag{5.1}$$

Now for the $N = 2$ case, the particular solution M of the Lax pair (1.7) and (1.8) is given by an invertible 2×2 matrix expressed in terms of column solutions $|m_1\rangle$ and $|m_2\rangle$: $M = (|m_1\rangle \quad |m_2\rangle)$. We take the 2×2 eigenvalue matrix Λ to be $\Lambda = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}$, where we have taken $\lambda_1 = \mu$ and $\lambda_2 = \bar{\mu}$. With this the solution $V[2]$ is written as

$$\begin{aligned}
\widetilde{V} &= \begin{vmatrix} M & I \\ M\Lambda & \boxed{\lambda I} \end{vmatrix} V = \left(\lambda I - \mu \frac{|m_1\rangle\langle m_1|}{\langle m_1|m_1\rangle} + \bar{\mu} \frac{|m_2\rangle\langle m_2|}{\langle m_2|m_2\rangle}\right) V \\
&= (\lambda I - \mu P - \bar{\mu} P^{\perp})V = (\lambda - \bar{\mu}) \left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} P\right) V,
\end{aligned}$$

where the Hermitian projection is $P = \frac{|m_1\rangle\langle m_1|}{\langle m_1|m_1\rangle}$, with the orthogonal projection $P^{\perp} = I - P = \frac{|m_2\rangle\langle m_2|}{\langle m_2|m_2\rangle}$. The Darboux matrix $D(\lambda)$ as a quasideterminant may be expressed in terms of the Hermitian projection and orthogonal projection as

$$D(\lambda) = \begin{vmatrix} M & I \\ M\Lambda & \boxed{\lambda I} \end{vmatrix} = (\lambda - \bar{\mu}) \left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} P\right) = (\lambda - \bar{\mu}) \left(P^{\perp} + \frac{\lambda - \mu}{\lambda - \bar{\mu}} P\right). \tag{5.2}$$

The one-soliton solution $\widetilde{g}$ in this case is given by

$$\widetilde{g} = \begin{vmatrix} M & I \\ M\Lambda & \boxed{O} \end{vmatrix} g = -\bar{\mu} \left(I + \frac{\mu - \bar{\mu}}{\bar{\mu}} P\right) g = -\bar{\mu} \left(P^{\perp} + \frac{\mu}{\bar{\mu}} P\right). \tag{5.3}$$

For the construction of explicit solution in the matrix form using the Darboux transformation, we take the example of $\mathcal{G} = SU(2)$. The solutions can be obtained by the Darboux transformation by taking the trivial solution as the seed solution. We have been considering the case where $j_{\pm} \in \mathbf{su}(2)$, the following discussions, however, are essentially the same for the Lie algebra $\mathbf{u}(2)$. Let us take a most general unimodular 2×2 matrix representing an element of the Lie algebra $\mathbf{su}(2)$,

$$\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix},$$

where Y and X are complex numbers satisfying $X\bar{X} + Y\bar{Y} = 1$. Let $j_{\pm}$ be the non-zero constant (commuting) elements of $\mathbf{su}(2)$, such that they are represented by anti-Hermitian 2×2 matrices

$$j_+ = \begin{pmatrix} \mathrm{i}p & 0 \\ 0 & -\mathrm{i}p \end{pmatrix}, \qquad j_- = \begin{pmatrix} \mathrm{i}q & 0 \\ 0 & -\mathrm{i}q \end{pmatrix}, \tag{5.4}$$

where p, q are non-zero real numbers. The seed solution is then written as

$$g(x^+, x^-) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(px^+ + qx^-)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}(px^+ + qx^-)} \end{pmatrix}. \tag{5.5}$$

The corresponding $V(\lambda)$ is

$$V(\lambda) = \begin{pmatrix} \omega(\lambda) & 0 \\ 0 & \omega^{-1}(\lambda) \end{pmatrix}, \tag{5.6}$$

where

$$\omega(\lambda) = \exp \mathrm{i} \left(\frac{1}{1-\lambda} px^+ + \frac{1}{1+\lambda} qx^- \right). \tag{5.7}$$

In this sense g, $j_\pm$, and V constitute the seed solution for the Darboux transformation. Taking $\lambda_1 = \mu$ and $\lambda_2 = \bar{\mu}$, we have the following 2×2 matrix solution of the Lax pair at $\Lambda = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{smallmatrix}\right)$

$$\begin{aligned} M &= (V(\mu)|1\rangle, V(\bar{\mu})|2\rangle) = (|m_1\rangle, |m_2\rangle), \\ &= \begin{pmatrix} \omega(\mu) & \omega(\bar{\mu}) \\ -\omega^{-1}(\mu) & \omega^{-1}(\bar{\mu}) \end{pmatrix}. \end{aligned} \tag{5.8}$$

The reality condition (2.30) on V implies that

$$\begin{aligned} \bar{\omega}(\bar{\mu}) &= \omega^{-1}(\mu), \\ \omega(\mu) &= \bar{\omega}^{-1}(\bar{\mu}). \end{aligned} \tag{5.9}$$

By direct calculations, we note that the S matrix in this case is given by

$$\begin{aligned} S &= M\Lambda M^{-1}, \\ &= \frac{1}{\mathrm{e}^r + \mathrm{e}^{-r}} \begin{pmatrix} \mu\,\mathrm{e}^r + \bar{\mu}\mathrm{e}^{-r} & (\bar{\mu} - \mu)\,\mathrm{e}^{\mathrm{i}s} \\ (\bar{\mu} - \mu)\,\mathrm{e}^{-\mathrm{i}s} & \bar{\mu}\,\mathrm{e}^r + \mu\mathrm{e}^{-r} \end{pmatrix}, \end{aligned} \tag{5.10}$$

where the functions $r(x^+, x^-)$ and $s(x^+, x^-)$ are defined by

$$\begin{aligned} r(x^+, x^-) &= i \left(\frac{1}{(1-\mu)} - \frac{1}{(1-\bar{\mu})} \right) px^+ + i \left(\frac{1}{(1+\mu)} - \frac{1}{(1+\bar{\mu})} \right) qx^-, \\ s(x^+, x^-) &= \left(\frac{1}{(1-\mu)} + \frac{1}{(1-\bar{\mu})} \right) px^+ + \left(\frac{1}{(1+\mu)} + \frac{1}{(1+\bar{\mu})} \right) qx^-. \end{aligned} \tag{5.11}$$

Let us take the eigenvalue to be $\mu = \mathrm{e}^{\mathrm{i}\theta}$. The expression (5.10) then becomes

$$S = \begin{pmatrix} \cos\theta + \mathrm{i}\sin\theta\tanh r & -\mathrm{i}(\sin\theta \operatorname{sech} r)\,\mathrm{e}^{\mathrm{i}s} \\ -\mathrm{i}(\sin\theta\operatorname{sech} r)\,\mathrm{e}^{-\mathrm{i}s} & \cos\theta - \mathrm{i}\sin\theta\tanh r \end{pmatrix}, \tag{5.12}$$

and the corresponding Darboux matrix $D(\lambda)$ in this case is

$$D(\lambda) = \begin{pmatrix} \lambda - \cos\theta - \mathrm{i}\sin\theta\tanh r & \mathrm{i}(\sin\theta\operatorname{sech} r)\,\mathrm{e}^{\mathrm{i}s} \\ \mathrm{i}(\sin\theta\operatorname{sech} r)\,\mathrm{e}^{-\mathrm{i}s} & \lambda - \cos\theta + \mathrm{i}\sin\theta\tanh r \end{pmatrix}. \tag{5.13}$$

Comparing the above equation with (4.3), we find the following expression for the projector,

$$P = \begin{pmatrix} 2\,\mathrm{e}^r\operatorname{sech} r & -2\,\mathrm{e}^{\mathrm{i}s}\operatorname{sech} r \\ -2\,\mathrm{e}^{-\mathrm{i}s}\operatorname{sech} r & 2\,\mathrm{e}^{-r}\operatorname{sech} r \end{pmatrix}, \tag{5.14}$$

which is the same as obtained in [4]. The solution $\widetilde{g}$ of chiral field equations is written as

$$\widetilde{g} = D(\lambda)|_{\lambda=0}\, g = -Sg, \tag{5.15}$$

$$= \begin{pmatrix} \widetilde{X} & \widetilde{Y} \\ -\overline{\widetilde{Y}} & \overline{\widetilde{X}} \end{pmatrix} g, \tag{5.16}$$

where the matrix entries are the functions

$$\widetilde{X} = -(\cos\theta + \mathrm{i}\sin\theta\tanh r), \tag{5.17}$$

$$\widetilde{Y} = \mathrm{i}(\sin\theta\,\mathrm{sech}\, r)\,\mathrm{e}^{\mathrm{i}s}. \tag{5.18}$$

The above expressions indicate that the functions $\widetilde{X}$ and $\widetilde{Y}$ have a solitonic form. Since we have

$$\widetilde{j}_\pm = (I-S)j_\pm(I-S)^{-1}. \tag{5.19}$$

Using equations (5.4) and (5.10) in the above equation, we get the expressions for $\widetilde{j}_\pm$ as

$$\widetilde{j}_+ = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad \widetilde{j}_- = \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix}, \tag{5.20}$$

where

$$a = \mathrm{i}p(1-(1+\cos\theta)\mathrm{sech}^2 r),$$
$$b = -\mathrm{i}p[(1+\cos\theta)\tanh r + \mathrm{i}\sin\theta](\mathrm{sech} r)\,\mathrm{e}^{\mathrm{i}s},$$
$$c = \mathrm{i}q(1-(1-\cos\theta)\mathrm{sech}^2 r),$$
$$d = \mathrm{i}q[(1-\cos\theta)\tanh r - \mathrm{i}\sin\theta](\mathrm{sech} r)\,\mathrm{e}^{\mathrm{i}s}.$$

Equation (5.20) shows a new solution which we have obtained by starting from an arbitrary seed solution. By substituting above expressions of a, b, c, d in (5.20), we see that $\mathrm{Tr}\,\widetilde{j}_+ = \mathrm{Tr}\,\widetilde{j}_- = 0$. Therefore, $\widetilde{j}_\pm$ satisfy the additional constraints for $g \in SU(2)$. Consequently, when we use the above equations in (5.20), we get the explicit expressions of the conserved currents (solutions) $\widetilde{j}_\pm$ of the chiral models by using the Darboux transformation.

The two-soliton solution of the chiral field is obtained by the application of the two-fold Darboux transformation

$$\widetilde{\widetilde{X}} = \frac{A+B}{\sin\theta_2\sin\theta_1(\sinh r_2\sinh r_1 - \cos(s_2-s_1)) - (1-\cos\theta_2\cos\theta_1)\cosh r_1\cosh r_2},$$
$$\widetilde{\widetilde{Y}} = \frac{C}{2[\sin\theta_2\sin\theta_1(\sinh r_2\sinh r_1 - \cos(s_2-s_1)) - (1-\cos\theta_2\cos\theta_1)\cosh r_1\cosh r_2]}, \tag{5.21}$$

where

$$A = \cos\theta_2\cosh r_2\cosh r_1 + \mathrm{i}\sinh r_2\sinh r_1(\sin\theta_2 - \sin\theta_1) - \mathrm{i}\sin\theta_2\sin^2\theta_1\sinh r_1\,\mathrm{sech}\, r_1$$
$$- \cos\theta_2(\cos\theta_1\cosh r_1 - \mathrm{i}\sin\theta_1\sinh r_1)(\cos\theta_2\cosh r_2 + \mathrm{i}\sin\theta_2\sinh r_2),$$
$$B = \sin\theta_2\sin\theta_1[(\cos\theta_1 - \mathrm{i}\sin\theta_1\tanh r_1)\,\mathrm{e}^{\mathrm{i}(s_1-s_2)}$$
$$+ (-2\cos\theta_2 + \cos\theta_1 + \mathrm{i}\sin\theta_1\tanh r_1)\,\mathrm{e}^{-\mathrm{i}(s_1-s_2)}]],$$
$$C = -\mathrm{i}\sin\theta_2\cosh r_1[1 - (\cos\theta_1 + \mathrm{i}\sin\theta_1\tanh r_1)(2\cos\theta_2 - \cos\theta_1 - \mathrm{i}\sin\theta_1\tanh r_1)]\,\mathrm{e}^{\mathrm{i}s_1}$$
$$+ \mathrm{i}\sin\theta_1\cosh r_2[1 + (\cos\theta_2 + \mathrm{i}\sin\theta_2\tanh r_2)(2\cos\theta_1 - \cos\theta_2 - \mathrm{i}\sin\theta_2\tanh r_2)]\,\mathrm{e}^{\mathrm{i}s_2}$$
$$+ \mathrm{i}\sin\theta_1\sin\theta_2\,\mathrm{e}^{\mathrm{i}s_1}(\sin\theta_2\mathrm{sech} r_2 - \sin\theta_1\,\mathrm{sech}\, r_1\,\mathrm{e}^{\mathrm{i}(s_1-s_2)}),$$

and we use the notation $X[3] = \widetilde{\widetilde{X}}$ and $Y[3] = \widetilde{\widetilde{Y}}$. We have generated a new solution by starting from an arbitrary seed solution. We can use the above equations (5.21) to find the expression for $S[2]$, which can be further used to obtain the explicit expressions of the conserved currents $j_\pm[3]$.

In the asymptotic limit for $t \to \pm\infty$, we have $r \to \pm\infty$, and equation (5.10) becomes

$$\lim_{r\to\pm\infty} S = \begin{pmatrix} \nu & 0 \\ 0 & \bar{\nu} \end{pmatrix}, \tag{5.22}$$

where

$$\begin{aligned} \nu &= \mu, \qquad \text{for} \quad r \to +\infty \\ &= \bar{\mu}, \qquad \text{for} \quad r \to -\infty. \end{aligned} \tag{5.23}$$

For $\mu = \mathrm{e}^{\mathrm{i}\theta}$, equation (5.22) becomes

$$\lim_{r\to\pm\infty} S = \begin{pmatrix} \mathrm{e}^{\pm \mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mp \mathrm{i}\theta} \end{pmatrix}, \tag{5.24}$$

and the functions $\widetilde{X}$ and $\widetilde{Y}$ in the solution $\widetilde{g}$ of the chiral model given by equations (5.17) and (5.18) become

$$\begin{aligned} &\lim_{r\to\pm\infty} \widetilde{X} = -\mathrm{e}^{\pm \mathrm{i}\theta} = -(\cos\theta \pm \mathrm{i}\sin\theta), \\ &\lim_{r\to\pm\infty} \widetilde{Y} = 0. \end{aligned} \tag{5.25}$$

The second iteration of the Darboux transformation can be used in a similar manner, and we have from equations (5.21)

$$\begin{aligned} &\lim_{r\to\pm\infty} \approx{X} = \exp \pm \mathrm{i}(\theta_2 + \theta_1) = (\cos(\theta_2 + \theta_1) \pm \mathrm{i}\sin(\theta_2 + \theta_1)) \\ &\lim_{r\to\pm\infty} \approx{Y} = 0. \end{aligned} \tag{5.26}$$

We see that in the asymptotic limit, we get much simpler expressions. Equation (5.26) gives the asymptotic behaviour of the solution $g[3]$ and it is clear from the above expression that in the asymptotic limit $g[3]$ i.e. the two-soliton solution splits into two single soliton solutions. Similarly for the Kth iteration of the Darboux transformation, the multisoliton solution in the asymptotic limit is given as

$$\lim_{r\to\pm\infty} g[K+1] = \lim_{r\to\pm\infty} \begin{pmatrix} X[K+1] & Y[K+1] \\ -\bar{Y}[K+1] & \bar{X}[K+1] \end{pmatrix} g, \tag{5.27}$$

where

$$\begin{aligned} \lim_{r\to\pm\infty} X[K+1] &= (-1)^K \exp \pm \mathrm{i}(\theta_K + \cdots \theta_1), \\ &= (-1)^K \left(\cos\left(\theta_K + \cdots \theta_1\right) \pm \mathrm{i}\sin(\theta_K + \cdots \theta_1)\right), \\ \lim_{r\to\pm\infty} Y[K+1] &= 0, \end{aligned} \tag{5.28}$$

which shows that the K-soliton solution $g[K+1]$ of the chiral model splits into K single solitons, where g is given by equation (5.5). Note that the $\pm$ sign appearing in the expression (5.28) due to $t \to \pm\infty$ shows that there is a phase shift in the soliton. Therefore, we see that when $t \to \pm\infty$, the asymptotic solution splits up into K single solitons.

From the above calculations we see that in the asymptotic limit $S[k] = M[k]\Lambda_k M[k]^{-1} \to M_k\Lambda_k M_k^{-1}$. Therefore in the asymptotic limit, the quasideterminant (3.11) splits into K factors, i.e.,

$$\begin{aligned} \lim_{r\to\pm\infty} g[K+1] &= \lim_{r\to\pm\infty} \begin{vmatrix} M_1 & M_2 & \cdots & M_K & I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & O \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & O \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{O} \end{vmatrix} g, \\ &= \begin{vmatrix} M_K & I \\ M_K\Lambda_K & \boxed{O} \end{vmatrix} \begin{vmatrix} M_{K-1} & I \\ M_{K-1}\Lambda_{K-1} & \boxed{O} \end{vmatrix} \cdots \begin{vmatrix} M_1 & I \\ M_1\Lambda_1 & \boxed{O} \end{vmatrix} g, \end{aligned}$$

$$= \prod_{k=1}^{K} (-1)^k \begin{vmatrix} M_{K-k+1} & I \\ M_{K-k+1}\Lambda_{K-k+1} & \boxed{O} \end{vmatrix} g. \tag{5.29}$$

We can say that the splitting of the K-soliton solution into K single soliton solutions asymptotically is in fact equivalent to the factorization of quasideterminant solution (3.11) into a product of quasideterminants of 2×2 matrices over a noncommutative ring R of $N \times N$ matrices.

6. Concluding remarks

In this paper, we have considered the principal chiral model in two dimensions, based on some Lie group, and presented the quasideterminant solutions of the chiral model as well as its Lax pair obtained by means of the Darboux transformation, defined in terms of the Darboux matrix. We iterated the Darboux transformation to get the quasideterminant multisoliton solutions. We have also discussed the relation of the Darboux matrix approach with the Zakharov–Mikhailov's dressing method, where the Darboux matrix was shown to be expressed in terms of the Hermitian projector defined in terms of particular column solutions of the Lax pair. At the end, we calculated the one- and two-soliton solutions for the case of Lie group $SU(2)$. The asymptotic limit of the solutions in the $SU(2)$ case splits the solution into product of single solitons. We have also obtained the asymptotic solution of the chiral model in terms of quasideterminant for the case of Lie group $SU(2)$. It would be interesting to study the quasideterminant solutions of the supersymmetric chiral models and those of the nonlinear sigma models based on symmetric spaces. We shall address these issues in a separate work.

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